Econ 204B: Section 4

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Announcements

- Midterm Exam: Thursday, February 7th
- A Practice Midterm (and solutions) will be posted on my website
- Extra Office Hours?

Solution Methods: Introduction

- Today we will see a few different ways of solving dynamic programs of the form we setup last time
- The first two (**guess and verify** and **value function iteration**) are directly solving the problem at hand (i.e. solving for the value / policy functions)
- The last "method" (**functional EE**) is actually a tool that is useful for finding the EE / allocations / steady state values (which are often of interest)
  - construct a Lagrangian out of the Bellman Equation (and constraints)
- But first, it might be useful to know where we are trying to go; one starting point is the concept of a **Recursive Competitive Equilibrium**

Recursive Competitive Equilibrium

Definition. Let K denote the state of the aggregate economy and a denote the personal state of an agent. Then a *Recursive Competitive Equilibrium (RCE)* is a set of functions that describe

Quantities: $K' = G(K)$, $a' = g(a, K)$ (agg. and personal policy functions)
Lifetime Utility: $V(a, K)$ (the value function)
Prices: $r(K) = f_K(K, N) + 1 - \delta$, $w(K) = f_N(K, N)$ (competitive prices)

such that:

1. Prices are complete and given
2. $V(a, K)$ and $g(a, K)$ solve the consumer's maximization problem
3. Consistency: $G(K) = g(K, K)$

That is, households must know prices so they can make utility maximizing decisions and if we gave on person all of the capital, their choice would coincide with the aggregation of everyone else's choices.

Guess and Verify

- This method is also known as the *Method of Undetermined Coefficients*
- It involves *guessing* the form of the value/policy function with some stand-in coefficients, and then *verifying* that the guess is consistent with the optimization problem
- The idea is that, if the guess is correct, then when "operated on" it should recover that same form and we can back-out the coefficients previously left *undetermined*
  - Recall our solution is a fixed point: $Tv^* = v^*$
- This solution technique has several requirements:
  - Unique Solution
  - "Correct Guess"

Guess and Verify: Policy Function

Suppose the household solves the following problem

$$U = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t) \qquad s.t. \qquad c_t + k_{t+1} = k_t^{\alpha}$$

The recursive problem is given by the following

$$V(k) = \max_{k'} \{\ln(k^{\alpha} - k') + \beta V(k')\}$$

Guess that $k' = \eta k^{\alpha}$. Solve for the undetermined coefficient, η, and find the policy function for capital and consumption

First take the F.O.C. of the optimization problem.

$$\frac{1}{k^\alpha - k'} = \beta \frac{dV(k')}{dk'}$$

Note that there is a k' on the LHS. The optimal policy function must satisfy the F.O.C, thus we plug in our guess.

$$\frac{1}{k^\alpha - (\eta k^\alpha)} = \beta \frac{dV(k')}{dk'} \qquad \Longrightarrow \qquad \frac{1}{(1-\eta)k^\alpha} = \beta \frac{dV(k')}{dk'}$$

Now, we can utilize the envelope theorem to find $dV(k')/dk'$.

$$\frac{dV(k)}{dk} = \frac{1}{k^\alpha - k'}(\alpha k^{\alpha-1}) = \underbrace{\frac{1}{(1-\eta)k^\alpha}(\alpha k^{\alpha-1})}_{\text{plug in the guess}} = \frac{\alpha}{(1-\eta)k}$$

"Pushing forward," we have

$$\frac{dV(k')}{dk'} = \frac{\alpha}{(1-\eta)k'} = \underbrace{\frac{\alpha}{(1-\eta)\eta k^{\alpha}}}_{\text{plug in the guess}}.$$

Putting this back into the F.O.C. we can now solve for the undetermined coefficient η.

$$\frac{1}{(1-\eta)k^{\alpha}} = \frac{\alpha\beta}{(1-\eta)\eta k^{\alpha}} \qquad \Longrightarrow \qquad \eta = \alpha\beta,$$

which we can verify is a constant. The policy functions for k' and c are thus

$$k' = \alpha\beta k^{\alpha} \qquad \text{and} \qquad c = (1-\alpha\beta)k^{\alpha}.$$

Guessing the Value Function

Now let's try our hand at guessing the value function for the same problem:

$$V(k) = \max_{k'} \left\{ ln(k^\alpha - k') + \beta V(k') \right\}.$$

In particular, guess that $V(k) = A + Bln(k)$, where A and B are the undetermined coefficients.

First let's solve the maximization problem on the RHS, plugging in our guess for $V(k')$.

$$\frac{dRHS}{dk'} = \frac{d}{dk'} \left\{ ln(k^\alpha - k') + \beta \left[A + Bln(k') \right] \right\} = 0$$

$$\implies \qquad k' = \frac{\beta B k^\alpha}{1 + \beta B}$$

Now, to evaluate the RHS at the optimum we plug in k'.

$$RHS\left(k'^*\right) = ln\left(k^\alpha - \frac{\beta B k^\alpha}{1+\beta B}\right) + \beta\left[A + Bln\left(\frac{\beta B k^\alpha}{1+\beta B}\right)\right]$$

$$= ln\left(\frac{k^\alpha}{1+\beta B}\right) + \beta A + \beta Bln\left(\frac{\beta B k^\alpha}{1+\beta B}\right)$$

Now, group the constants together and the k terms separately.

$$RHS\left(k'^{*}\right) = \underbrace{-ln(1+\beta B) + \beta A + \beta B ln\left(\frac{\beta B}{1+\beta B}\right)}_{\text{constant}} + \underbrace{\alpha(1+\beta B)}_{ln(k)\text{-term coeff.}} ln(k)$$

We are almost there. Now recall that the V on the LHS will also have the form of our guess. This is to say that we know the result above is *equal to* $A + Bln(k)$ (note also that we can verify whether the guess was right).

$$A = \text{constant} = -ln(1+\beta B) + \beta A + \beta B ln\left(\frac{\beta B}{1+\beta B}\right)$$

$$B = ln(k)\text{-term coeff.} = \alpha(1+\beta B)$$

Now solve for B using the second equation, and then A from the first.

$$B = \frac{\alpha}{1 - \alpha\beta} \qquad \text{and} \qquad A = \frac{1}{1-\beta}\left[ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} ln(\alpha\beta) \right]$$

We might also want to get the policy function. Recall that we actually solved for it earlier. All we need to do is plug in for B.

$$k' = \frac{\beta B k^{\alpha}}{1 + \beta B} \qquad \Longrightarrow \qquad k' = \alpha\beta k^{\alpha}$$

Value Function Iteration

- This method relies on the ideas of a contraction / fixed point that we discussed last time
- Recall: (*paraphrasing*) "starting from any possible V_0 in the space of potential solutions, when iterating on V_0 we will get closer and closer to the true V"
- That is, we are going to make an initial guess at V (a good place to start is $V_0 = 0$), and then operate for a while until we are comfortable that the value function has converged

- The above is true if we are assuming an infinite horizon; what if there is some terminal period T (for example, a lifecycle model)?
- Then you can think of this process like you would backwards induction...
  - V_0 is the value at $T + 1$ $(= 0)$
  - V_1 is the value you get in T when optimizing knowing that $V_0 = 0$
  - etc.
- You would continue this process until you found the value function for the first date through the last date: $V_T, \ldots, V_1, V_0$, and your answer would effectively be the sequence of value (or policy) functions
- Let's try our hand at an easy example; consider the problem we've been working with and initialize the iterations with $V_0(k') = 0$

First Iteration

$$V_1(k) = \max_{k'} \left\{ ln(k^\alpha - k') + \beta \underbrace{0}_{V_0(k')} \right\}$$

Optimization requires that $k' = g_1(k) = 0$ (calculus doesn't work here, we are at a "corner"). Knowing this, we can plug in this optimal policy to find $V_1(k)$.

$$V_1(k) = ln(k^\alpha - 0) = \alpha ln(k)$$

Now for a second round ...

Second Iteration

$$V_2(k) = \max_{k'} \left\{ ln(k^\alpha - k') + \beta \underbrace{\alpha ln(k')}_{V_1(k')} \right\}$$

Let's optimize.

$$\frac{dV_2(k)}{dk'} = 0 \qquad \Longrightarrow \qquad k' = g_2(k) = \frac{\alpha\beta}{1+\alpha\beta} k^\alpha$$

Knowing this, we can plug in this optimal policy to find $V_2(k)$.

$$V_2(k) = ln\left(\frac{k^\alpha}{1+\alpha\beta}\right) + \alpha\beta ln\left(\frac{\alpha\beta}{1+\alpha\beta} k^\alpha\right)$$

$$= ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta) ln(k)$$

Let's go for a third, round, paying attention to an emerging pattern.

Third Iteration

$$V_3(k) = \max_{k'} \left\{ ln(k^\alpha - k') + \beta \underbrace{\left[ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta) ln(k') \right]}_{V_2(k')} \right\}$$

Optimize.

$$\frac{dV_3(k)}{dk'} = 0 \qquad \Longrightarrow \qquad k' = g_3(k) = \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} k^\alpha$$

Plug the optimal policy back in to find $V_3(k)$.

$$V_3(k) = \beta ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta^2 ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + (\alpha\beta + (\alpha\beta)^2) ln\left(\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right)$$

$$+\alpha(1+\alpha\beta+(\alpha\beta)^2) ln(k)$$

We could continue (or make our computer do it), but for this problem we can see a pattern emerge. Perchance most nobaly on the policy function, we can see that as we let the iteration $s \to \infty$, we'll have

$$g^*(k) = \lim_{s \to \infty} g_s(k) = \alpha\beta k^\alpha.$$

See earlier slides for how to "derive" the above. The Value function itself can be shown to converge to

$$V^*(k) = \lim_{s \to \infty} V_s(k) = \frac{1}{1-\beta}\left[ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}ln(\alpha\beta)\right] + \frac{\alpha}{1-\alpha\beta}ln(k),$$

which is, recall, what we found earlier with *Guess and Verify*.

Functional Euler Equation

- This last "method" doesn't actually solve for the value / policy functions directly (though you can back them out)
- Sometimes you might see this called the Euler-Lagrange Equation
- It involves the construction of a "Lagrangian" using the Bellman Operator; here, though, we are mapping functions to functions (hence "functional equation")
- The procedure should feel very familiar to you, as you've sort of seen it in previous sections; set up the "Lagrangian" and take F.O.C.s, find the EE and use constraints / conditions (e.g. market clearing) to solve for whatever you desire

Just to get an idea, let's try it out on our running example:

$$\mathcal{L} = ln(c) + \beta V(k') + \lambda\big[k^\alpha - c - k'\big]$$

$$\frac{\partial \mathcal{L}}{\partial c} = 0: \qquad \lambda = \frac{1}{c} \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial k'} = 0: \qquad \lambda = \beta \frac{dV(k')}{dk'} \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0: \qquad c + k' = k^\alpha \tag{3}$$

Combine (1) and (2).

$$\frac{1}{c} = \beta \frac{dV(k')}{dk'} \tag{4}$$

Now let's use the envelope theorem. Evaluating $\mathcal{L}$ at the same value as V on the RHS, the two are equal. Further we know that $V \leq \mathcal{L}$. Thus

$$\frac{\partial \mathcal{L}}{\partial k} = \frac{dV(k)}{dk} \qquad \Longrightarrow \qquad \frac{dV(k)}{dk} = \lambda[\alpha k^{\alpha-1}]$$

Pusing this forward, and plugging in for $\lambda' = 1/c' \ldots$

$$\frac{dV(k')}{dk'} = \frac{1}{c'}[\alpha k'^{\alpha-1}], \tag{5}$$

noting that $\alpha k^{\alpha-1} = 1 + r$ when $\delta = 1$ (which is what we have in this example). Now, plug (5) back into (4) to reunite with an old friend.

$$\frac{1}{c} = \beta[\alpha k'^{\alpha-1}]\frac{1}{c'} \tag{EE}$$

- From here we can do many things. . .
- You might want to find steady state values: plug in to budget constraints / market clearing conditions, do comparative statics, etc.
- Alternatively, you may want to recover the policy function: one way would be to iterate on the EE like we did in section 2 (you would find $k' = \alpha\beta k^{\alpha}$)
- This technique may seem more roundabout than last time (recall I just plugged the constraint right into the objective), but just note that this method is slightly more general insofar as it handles situations where you can't easily substitute in all constraints