Neoclassical Growth Model

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Recap

- We have seen how to set-up the Hamiltonian and apply the Maximum Principle
  - Yields 2 differential equations
  - Require 2 terminal conditions
- Must satisfy two conditions to pin down optimal initial consumption
  - No-ponzi scheme $\Rightarrow$ $k(T) \geq 0$
  - Transversality condition $\Rightarrow$ $k(T) \leq 0$
- We now turn to comparative statics, dynamics, and convergence of this model
- Here, we rely on phase diagrams rather than a single differential equation as in Solow

Steady State

- First, let's take our eqations which describe the dynamics of c and k in this economy (what we get after setting up the hamiltonian, applying the maximum principle, etc.)

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho - \theta g}{\theta} \qquad\qquad \dot{k} = f(k) - c - (\delta + n + g)k$$

- To impose the steady state, recall that efficiency units are constant $\Rightarrow$ that we need $\dot{k} = \dot{c} = 0$

$$\underbrace{f'(k^*) = \delta + \rho + \theta g}_{\dot{c}=0} \qquad\qquad \underbrace{c = f(k) - (\delta + n + g)k}_{\dot{k}=0}$$

- Because we need both to be satisfied simultaneously, it's useful to graph these loci

The $\dot{c} = 0$ Locus

$$f'(k^*) = \delta + \rho + \theta g \qquad \frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho - \theta g}{\theta}$$

- this curve directly pins down k^*
- because of DMR ($f''(k) < 0$), we can determine . . .
- if k low $\rightarrow f'(k)$ high $\rightarrow \dot{c} > 0$
- if k high $\rightarrow f'(k)$ low $\rightarrow \dot{c} < 0$

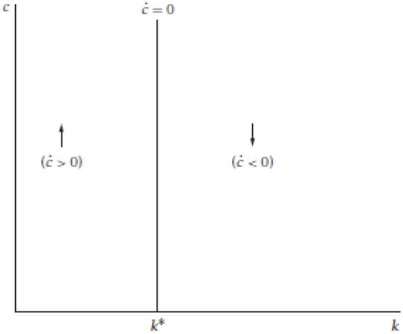


FIGURE 2.1 The dynamics of c

The $\dot{k} = 0$ locus

$$c = f(k) - (\delta + g + n)k \qquad \dot{k} = f(k) - c - (\delta + g + n)k$$

- fix in your mind some level of k ...
- if c high $\rightarrow \dot{k} < 0$
- if c low $\rightarrow \dot{k} > 0$
- the point at which this sign switches denotes when $\dot{k} = 0$
- we have already graphed that cutoff!

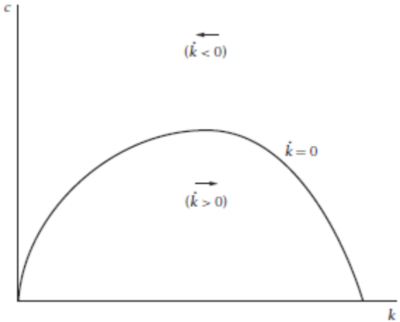


FIGURE 2.2 The dynamics of k

Key Graph

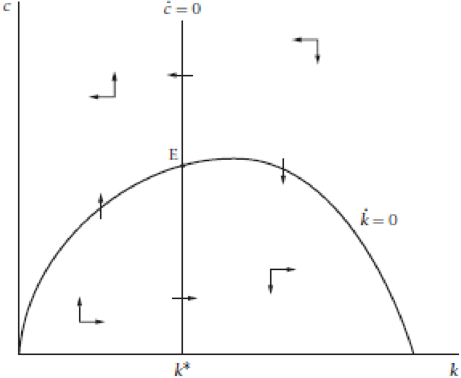


FIGURE 2.3 The dynamics of c and k

Visualizing What We've Done Previously

- given $k(0)$ we solved the diff eqs, applied terminal conditions, found $c(0)$ ...
- this procedure puts us on the optimal path (the "saddle path") that max's our objective
- infinite horizon: saddle path will resemble $F \rightarrow E$ and the correct choice of $c(0)$ will mean the economy converges to the steady state
- **Pop Quiz**: what would the optimal path look like for a finite horizon problem?

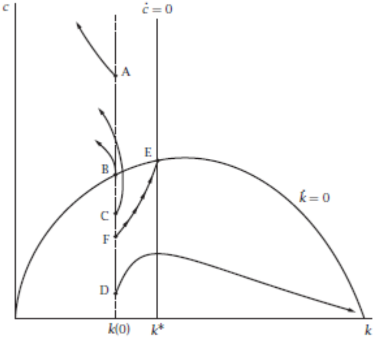


FIGURE 2.4 The behavior of c and k for various initial values of c

Saddle Path

- The **saddle path** is the route taken on the phase diagram that satisfies all constraints and optimality conditions.

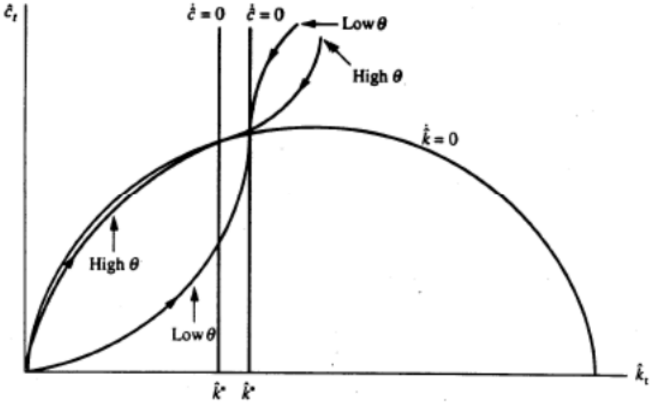
- Notice that there is a unique c^* for all k^*
  - Why existence?
  - Why uniqueness?

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho - \theta g}{\theta} \qquad \dot{k} = f'(k) - c - (\delta + n + g)k$$

- Saddle path is very different for differing preferences:
  - High IES $\Rightarrow$ fast convergence
  - Low IES $\Rightarrow$ slow convergence

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho - \theta g}{\theta}$$

Saddle Path



$\hat{c}_t$
$\dot{c} = 0$
$\dot{c} = 0$
Low θ
High θ
$\dot{\hat{k}} = 0$
High θ
Low θ
$\hat{k}^*$
$\hat{k}^*$
$\hat{k}_t$

Proof

- A natural question to ask is when is such an economy dynamically efficient vs. dynamically inefficient
  - **Dynamically Efficient**: No Pareto improving steps to be taken; some must suffer
  - **Dyanmically Inefficient**: Pareto improving steps can be take; increase consumption for all

- From the $\dot{k} = 0$ locus,

$$\frac{dc}{dk} = f'(k) - \delta - n - g \qquad \Rightarrow \qquad f'(k^{gr}) = \delta + n + g$$

- The $\dot{c} = 0$ curve is given by

$$f'(k^*) = \delta + \rho + \theta g$$

Proof

- Under the assumption of CRRA,

$$U = \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\theta}}{1-\theta} \right) \left(A(0) e^{gt} \right)^{1-\theta} L(0) e^{nt}$$

- Hence, we have implicitly assumed throughout these problems that $\rho - n - (1-\theta) g > 0$
- Now, let's compare capital:

$$\begin{aligned} &\rho > n + (1-\theta) g && \text{(Assumption)} \\ \Leftrightarrow &\rho + \delta + \theta g > \delta + n + g && \text{(Rearrange)} \\ \Leftrightarrow &f'(k^*) > f'(k^{gr}) && \text{(Def.)} \\ \Rightarrow &k^* < k^{gr} && (f''(\cdot) < 0) \end{aligned}$$

- Thus, we know that this economy is always dynamically efficient

Example Problem: CA Prop 64 Passes

Suppose that California Proposition 64 passes in the last year's November election (the marijuana / "wacky tobacky" one) that goes into affect next year. In an infinite horizon Ramsey economy, describe graphically what happens to k over time using the phase diagram for the following two different scenarios of this legislation that might play out. Assume that the economy starts out in the steady state.

(a) Suppose that this results in a discrete drop in A.

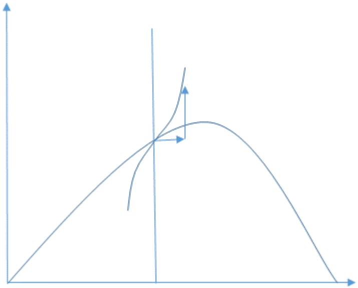
(b) Suppose, instead, that people become very anxious and ρ increases.

Change in A

(a) Suppose that this results in a discrete drop in A.

The drop in A doesn't change any fundamental parameters in the economy. Thus, neither locus changes. $k = K/AL$ will jump up, however.

At the higher level of k, the economy will want to increase its consumption immediately to put itself on the saddle path to return to the steady state. That is, the Ramsey model is consistent with "the munchies."

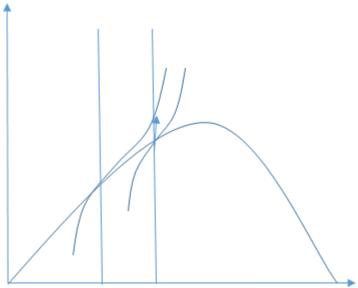


Change in ρ

(b) Suppose, instead, that people become very anxious / impatient and ρ increases.

In this scenario, we will again see the presence of the munchies. Looking at the expressions for the $\dot{c} = 0$ and $\dot{k} = 0$ curves from before, we see that ρ only appears (and will only shift) the former.

$$f'(k^*) = \delta + \rho + \theta g$$



Because of DMR, the increase in ρ will require that k^* decreases in order to maintain that equality (i.e. $\dot{c} = 0$ shifts to the left). When this happens, the new saddle path must lie above the old one, and optimizing agents will increase their consumption to place the economy on the saddle path.