Econ 204C: Section 6

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Heterogeneous Agents Models

- The starting point for study here are **Bewley Models**
  - agents are *ex ante* identical
  - they end up *ex post* heterogeneous because of idiosyncratic shocks
  - incomplete markets prevent sharing these risks
- Two main types within this class of models are **Huggett Models** (endowment) and **Aiyagari Models** (production)
  - *Huggett*: assets are not productive (think of claims); in the aggregate they are held in zero net supply
  - *Aiyagari*: there is a production side to the market; assets are held in positive net supply in the aggregate (the capital stock)

- While these models do a decent job of reconciling some *stylized facts*, there are still some shortfalls
- Some of these shortfalls have associated model extensions which (to various degrees) to account for them; for example . . .
- **Krusell & Smith**: adds aggregate uncertainty to the Aiyagari Model
  - computational finding: the mean is enough to track distribution and form expectations
- **Benhabib, Bisin, & Zhu**: Pareto distribution for wealth

Borrowing Limits

- A key feature of these models, indeed one of the things that makes them interesting, is that markets are *incomplete*
- With *complete* markets, the allocation depends on the current value of the state variables, reflecting the fact that agents have every opportunity to insure themselves against risk
- With *incomplete* markets, there will be impediments to exchanging risks that make allocations history dependent
- Starting with the basics, histories will be artifacts of random variables (usually iid or Markov) and market incompleteness (captured by **borrowing limits**)

Natural and *Ad Hoc* Borrowing Constraints

We often times assume that consumption has to be positive. This is implicitly the case if we assume that the utility function satisfies the Inada conditions. In any regard, the budget constraint of the household's problem will resemble

$$c_t = (1+r)a_t + ws_t - a_{t+1} \geq 0$$
$$\iff \quad a_t \geq \frac{1}{1+r}(a_{t+1} - ws_t),$$

where s_t is some finite state random variable that is Markov. For example, you can think of $s_t \in \{0, 1\}$ as signifying unemployed and employed states.

Since that relationship holds $\forall t$, we can write a similar expression for a_{t+1} and plug it in . . .

$$a_t \geq \frac{1}{1+r}\left[\frac{1}{1+r}(a_{t+2} - ws_{t+1}) - ws_t\right]$$

We could continue to do this iteratively. Write the expression for a_{t+2} and plug in, then for a_{t+3} .... Suppose we did this until some period T. We'd have

$$a_t \geq \frac{a_{t+T+1}}{(1+r)^{T+1}} - \frac{1}{1+r} \sum_{j=0}^{T} \frac{ws_{t+j}}{(1+r)^j}.$$

Since this model has an infinite horizon, we will take a limit as $T \to \infty$. The transversality condition (which is an optimality condition of the household's problem) makes the first term tend to 0. We are left with

$$a_t \geq -\frac{1}{1+r} \sum_{j=0}^{\infty} \frac{ws_{t+j}}{(1+r)^j}.$$

Remembering that this object tells us what our assets today can be as a function of our future income (which is a random variable), we now can start thinking about different borrowing limits.

Let's impose a condition that ensures that everyone will be able to pay back their debt. That is, even if somebody were to be very unlucky and get shocks $s_t = s_{min}$ every period, we want them to be able to pay back their debt.

$$\begin{aligned}\phi &= \frac{1}{1+r}\sum_{j=0}^{\infty}\frac{ws_{min}}{(1+r)^j} \\ &= \frac{ws_{min}}{1+r}\left(\frac{1}{1-\frac{1}{1+r}}\right) \\ &= \frac{ws_{min}}{r}\end{aligned}$$

This thing, ϕ, is referred to as the **natural borrowing limit**. It is the least restrictive limit that still ensures that everyone can pay back their debts.

Still yet, we may want to be even more stringent than that. To accommodate this, you'll sometimes see the borrowing constraint written as

$$\phi = \min\left\{b, \frac{ws_{min}}{r}\right\},$$

where the parameter $b > 0$ denotes an **ad hoc borrowing limit**.

A Savings Problem

Let there be a unit measure of households that maximize the present discounted value of lifetime utility over consumption.

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + a_{t+1} = (1+r)a_t + ws_t \quad \forall t$$
$$a_t \geq -\phi \quad \forall t$$

β is the discount factor and $u(c_t)$ is an increasing, strictly concave, twice continuously differentiable utility function that satisfies the Inada conditions.

s_t, for example, is some idiosyncratic finite state Markov shock with transition probabilities denoted by $\mathcal{P}(i,j) \equiv Prob(s' = s_j | s = s_i)$.

Suppose that we have realized a state s_i today. The value function for a household can be written as

$$V(a, s_i) = \max_{a'} \left\{ u\Big((1 + r)a + ws_i - a'\Big) + \beta \sum_{j=1}^{m} V(a', s_j)\mathcal{P}(i, j) \right\}$$
$$\text{s.t.} \quad a' \geq -\phi.$$

It will be important to track the evolution of the distribution of (a, s) over time, which we'll denote with $\lambda_t(a, s)$ (indeed, it's often interesting in and of itself).

The Markov transition probabilities and the policy function (assume that we have it), $a' = g(a, s)$, will induce a law of motion for the distribution λ_t. Let $\mathcal{A}$ denote the support of a and, for simplicity of presentation, assume that it's also finite.

$$\lambda_{t+1}(a', s') = \sum_{a \in \mathcal{A}} \sum_{s} \lambda_t(a, s) Prob(s_{t+1} = s' | s_t = s) \mathbb{I}(a', s, a)$$

The indicator defines $\mathbb{I}(a', a, s) = 1$ if $a' = g(a, s)$ and 0 otherwise (recall that this is a distribution, and so must specify a probability for all points in the support of a', and all of those not.

If we are reasonably clever, we can combine $Prob(\cdot)$ and $\mathbb{I}(\cdot)$ (and fiddle with the indexing) in the above expression using the Markov transition matrix:

$$\lambda_{t+1}(a', s') = \sum_{s} \sum_{\{a : a' = g(a,s)\}} \lambda_t(a, s) \mathcal{P}(s, s').$$

So, now we have specified the problem (for households) and have come up with a way to express the joint distribution (and how it evolves over time) of (a, s). It might be useful to know where we are going.

We can either define a production side (Aiyagari) where we will be able to determine the prices r and w in general equilibrium, or perhaps just work with an endowment economy (Huggett), to name to possibilities.

The equilibrium concept we are interested in is a *Stationary Recursive Competitive Equilibrium*, which will require that the distribution $\lambda_t = \lambda^*$ is not moving over time.

Luckily, by this point, we know something about stationary distributions (recall the Markov chain section). We know that if a finite state Markov chain is irreducible (it is possible to get to any state from any state), then there exists a unique stationary distribution.

Change of Variables

- It is sometimes useful to define a change of variables in these problems
- Typically this is done in order to reduce the number of state variables (which can simplify computational aspects of the problem)
- A very common change of variables is to define "disposable resources available" (sometimes "cash-in-hand") as

$$x_t \equiv (1+r)a_t + ws_t + \phi$$

- We can also write $\widehat{a}_t \equiv a_t + \phi \geq 0$; thus, for the problem at hand, the households budget set can be represented as

$$\begin{aligned} c_t + \widehat{a}_{t+1} &= x_t \\ x_{t+1} &= (1+r)\widehat{a}_{t+1} + ws_{t+1} - r\phi \end{aligned}$$