Econ 204A: Section 2

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Notes on HW 1

- Answer the full problem, even if it's just a guess.
- Think about assumptions. Are they needed? What do they buy us?
- Recall, the posted solutions to 1.6 solve the problem using the typo.

Solow Analysis: Comparative Statics

- The first thing we might concern ourselves with is how our equilibrium changes when model parameters change

- We should (almost) always start with the equilibrium condition and take a total derivative to isolate the effect the change has on k^* (of which all other quantities are integrally related)

$$sf(k^*) = (\delta + n + g)k^*$$

$$f(k^*)ds + sf'(k^*)dk^* = (\delta + n + g)dk^* + (d\delta + dn + dg)k^*$$

- From here we can pick-and-choose which statics to look at (and then try to "sign" them)

- Let's walk through an example where we determine how k^*, y^*, and c^* are affected by a change in s

- The first thing we'll need to do is isolate the changes we want using the total derivative.

$$f(k^*)ds + sf'(k^*)dk^* = (\delta + n + g)dk^* + (0 + 0 + 0)k^*$$

$$\implies \qquad \frac{dk^*}{ds} = \frac{f(k^*)}{(\delta + n + g) - sf'(k^*)} > 0$$

- Now, we can "build up" to determine the effects on y^* and c^*.

$$y^* = f(k^*) \qquad \implies \qquad \frac{dy^*}{ds} = f'(k^*)\frac{dk^*}{ds} > 0$$

$$c^* = (1 - s)y^* \qquad \implies \qquad \frac{dc^*}{ds} = (1 - s)\frac{dy^*}{ds} - y^* \gtreqless 0$$

The Golden Rule and Dynamic Efficiency

- The closest notion of welfare here in the Solow model is consumption
  - we don't really care about output or capital insofar as they enable us to consume
- If we have the iconic Solow diagram in mind, we can see that if the s were really low/high then the intersection of $sf(k)$ and $(\delta + n + g)k$ will be very close to $f(k)$ (i.e. consumption is low)
- Because of the shape of our curves, there is some middle ground where consumption is maximized
- That level is called the **Golden Rule** level of k
- It's the level you would have wanted people in the past to have chosen

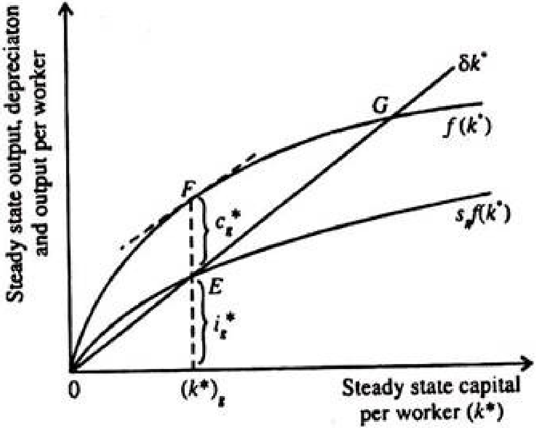


Fig. 4.8 The Saving Rate and the Golden Rule

- How will we determine k^{gr}? Well, we know that it will be chosen such that $dc/dk = 0$

$$c = f(k) - (\delta + n + g)k \qquad \Longrightarrow \qquad \frac{dc}{dk} = f'(k) - (\delta + n + g) = 0$$

$$f'(k^{gr}) = (\delta + n + g)$$

- I.e. it is the level of k such that the output curve is tangent to the break-even line

- Under the usual Cobb-Douglas specification with capital's share $= \alpha$, we'll have

$$k^{gr} = \left(\frac{\alpha}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}$$

- Compare the Golden rule level of k with the general steady state value of k we found last week:

$$k^{gr} = \left(\frac{\alpha}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} \qquad \text{vs.} \qquad k^{*} = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}$$

- Immediately we can see that $k^* = k^{gr}$ if $s = \alpha$. That is, we are saving in the same rate as capital's return per unit of output.

- If the economy is in a position where $s > \alpha$ (i.e. we are saving more than what we are getting back in capital returns), then the economy is said to be **dynamically inefficient**

- In such a scenario, it would be a Pareto improvement to immediately lower s (allowing greater consumption right away and into the future)

- Is this true when $s < \alpha$? No. An increase in s would be required which has negative effects on consumption today (though c will be higher in the LR)

Capital's Share: $\alpha(k)$

- Something that will be very useful, especially when we start looking at dynamics in the Solow model, will be this notion of capital's share
- Generally, we define capital's share as the income generated by capital divided by total income

$$\alpha(k) = \frac{f'(k)k}{f(k)} = \frac{\text{price of k} \times k}{\text{total income}} \in (0, 1)$$

- In competitive environments (like what we have here), inputs are paid their marginal products, meaning capital owners are paid $f'(k)$ for each unit of k supplied
- With Cobb-Douglas ($f(k) = k^{\alpha}$), we easily show that $\alpha(k) = \alpha$, meaning we have constant capital's shares
- Historically, capital's share of output was constant, so this is (was?) a reasonable assumption

- To give an example of its usefulness, let's actually prove a result we used earlier
- Show that the slope of the investment curve is less than the slope of the break-even investment line around the steady steady state . . .

$$sf'(k^*) - (\delta + n + g) < 0$$

$$s\frac{\alpha(k^*)f(k)}{k^*} - (\delta + n + g) < 0 \qquad \text{(using the definition of } \alpha(k))$$

$$s\frac{\alpha(k^*)f(k)}{k^*} - \frac{sf(k^*)}{k^*} < 0 \qquad \text{(using } sf(k^*) = (\delta + n + g)k^*)$$

The last line follows because $\alpha(k) \in (0, 1)$

Solow Analysis: Dynamics

- Here we will be concerned about how our variables (written in various ways) respond over time to changes to . . .
  - parameters $(s, \delta, n, g, \alpha)$
  - quantities (A, L, K)
- There are a lot of different quantities that we might be interested in learning about . . .
  - variables in efficiency units (k, y, c)
  - per capita variables $(K/L, Y/L, C/L)$
  - aggregate variables (K, Y, C)

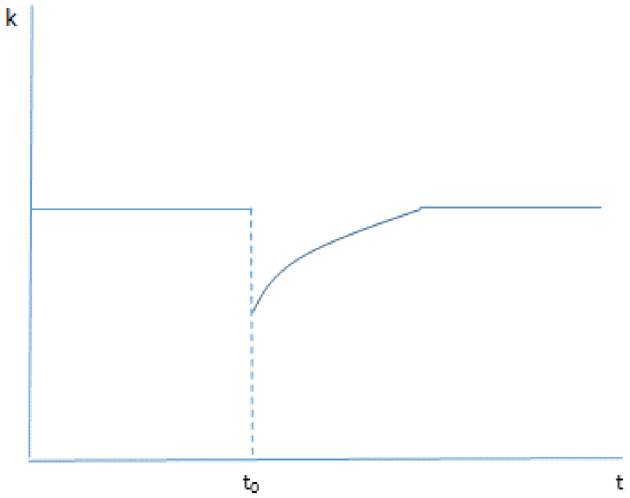
- While graphing the dynamics for the first category is easy (recall, these are stationary), the latter two quantities will be growing exponentially
- To get around this, we will graph the logs of those quantities so that the growth (in logs) is linear
- There are a lot of different things to look at / places to start
- As a good rule of thumb, always think about capital first (output and consumption are derived from capital)

Efficiency Units

- First consider a one-time change in an exogenous quantity: suppose that A doubles
  - because $k = K/AL$, we can see that there will be a discrete drop in k
  - when k drops discretely, $y = f(k)$ will drop discretely (f is increasing)
  - when y drops discretely, $c = (1 - s)y$ will drop discretely
- Over time, however, none of the fundamental parameters have changed (the steady state value will still be the same). Consider the dynamics of k

$$\dot{k} = sf(k) - (\delta + n + g)k > 0 \text{ when } k < k^*$$

- Takeaway: a change in an aggregate quantity will discretely affect our efficiency unit variables, but they will return to the same steady state in the LR


k
t_0
t

- Next consider a change in a model parameter: suppose that the rate of population growth decreases from n_0 to n_1
- Recall from class that K does not immediately respond (to anything), and so $k = K/AL$ will not discretely drop
- We do know that there will be a new, higher s.s. level for k (see comparative statics section).
- To determine what will happen to k in the meantime, consider looking at the growth rate of k just before and just after the change

$$\left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = \frac{sf(k)}{k} - (\delta + n_0 + g) \qquad \left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} = \frac{sf(k)}{k} - (\delta + n_1 + g)$$

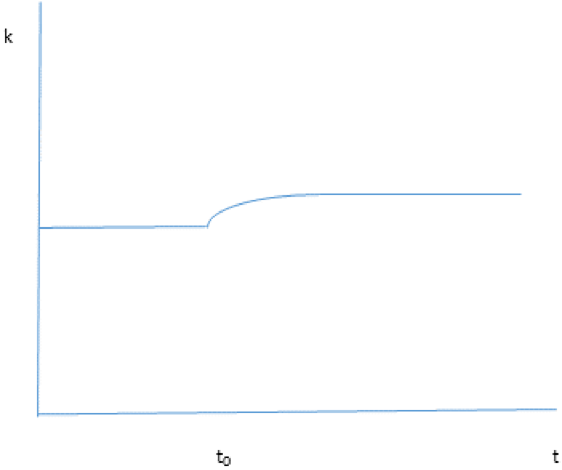
where the change occurs at time t_0 and $\varepsilon \to 0^+$

- We can now take a difference in these two growth rates

$$\left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} - \left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = n_0 - n_1 > 0,$$

which means that capital will be growing just after the change and this growth gets smaller until the new steady state is reached

- Takeaway: a change in a parameter will typically not lead to a discrete jump (consider a change in s on consumption); we have to be careful about what that growth looks like in the interim (is there a "kink"?)


k
t0
t

Per Capita Variables

- As noted earlier, per capita (and aggregate) variables are a little trickier because they are growing exponentially
- Recall that we can write per capita variables as follows

$$\frac{K}{L} = Ak \qquad \frac{Y}{L} = Ay = Af(k) \qquad \frac{C}{L} = Ac,$$

where (in the s.s.) k, y, and c aren't growing. A is growing at rate g, however, so these per capita variables are also growing at rate g in the s.s.

- Now, let's consider an example: suppose that g increases from g_0 to g_1 on capital per person and output per person

- The first thing that we'll want to do is take logs of the appropriate variables and then take time derivatives

$$ln(K/L) = ln(A) + ln(k) \qquad \implies \qquad \frac{\dot{K/L}}{K/L} = g + \frac{\dot{k}}{k}$$

$$\begin{aligned} ln(Y/L) = ln(A) + ln(f(k)) \qquad \implies \qquad \frac{\dot{Y/L}}{Y/L} &= g + \frac{f'(k)}{f(k)}\dot{k} \\ &= g + \frac{f'(k)k}{f(k)}\frac{\dot{k}}{k} \\ &= g + \alpha\frac{\dot{k}}{k} \end{aligned}$$

- Note the clever trick to back out capital's share, α

- Now we've got expressions for the dynamics of both quantities of interest; now all we need to do is determine how k is growing (same procedure as we did before)

$$\left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = \frac{sf(k)}{k} - (\delta + n + g_0) \qquad \left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} = \frac{sf(k)}{k} - (\delta + n + g_1)$$

$$\left.\frac{\dot{k}}{k}\right|_{t_0+\varepsilon} - \left.\frac{\dot{k}}{k}\right|_{t_0-\varepsilon} = g_0 - g_1$$

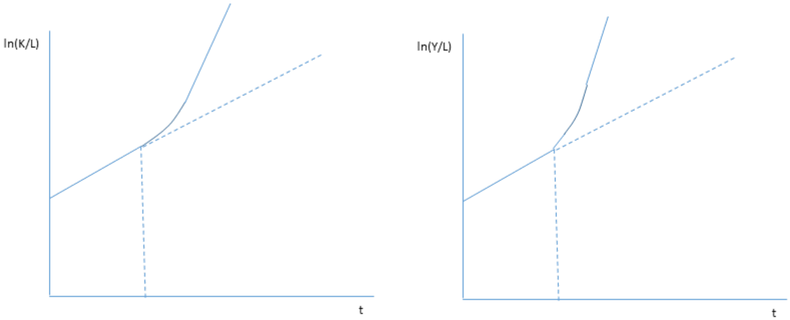
Per-Capita

- We know that per-capita variables grow at the rate of productivity $\Rightarrow$ steady state growth will be lower
- Now, use the same trick as with efficiency units to determine the transition path

$$\left.\frac{\dot{K/L}}{K/L}\right|_{t_0+\varepsilon} - \left.\frac{\dot{K/L}}{K/L}\right|_{t_0-\varepsilon} = g_1 + (g_0 - g_1) - g_0 = 0 \qquad \text{(No assumptions)}$$

$$\left.\frac{\dot{Y/L}}{Y/L}\right|_{t_0+\varepsilon} - \left.\frac{\dot{Y/L}}{Y/L}\right|_{t_0-\varepsilon} = g_1 + \alpha(g_0 - g_1) - g_0 > 0 \qquad \text{(Assuming } \alpha < 1\text{)}$$

- Hence, we have a kink in per-capita ouput, but no kink in per-capita capital


ln(K/L)
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