ECONOMICS 241B
2020 241B FINAL EXAMINATION

1. Let X and Y be two random variables with joint density:

$$f_{X,Y}(x,y) = \frac{3}{2}\left(x^2+y^2\right) \quad 0 \le x \le 1 \quad 0 \le y \le 1.$$

a. Derive the marginal densities, $f_X(x)$ and $f_Y(y)$.

Answer. The marginal densities of X and Y are

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{3}{2}\left(x^2+y^2\right)dy = \frac{3}{2}\int_0^1 \left(x^2+y^2\right)dy \\ &= \frac{3}{2}\left[x^2 y + \frac{1}{3}y^3\right]_0^1 = \frac{3}{2}\left(x^2+\frac{1}{3}\right), \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{3}{2}\left(x^2+y^2\right)dx = \frac{3}{2}\int_0^1 \left(x^2+y^2\right)dx \\ &= \frac{3}{2}\left[y^2 x + \frac{1}{3}x^3\right]_0^1 = \frac{3}{2}\left(y^2+\frac{1}{3}\right). \end{aligned}$$

b. Consider the linear projection

$$Y_i = \alpha + \beta X_i + \varepsilon_i.$$

Derive the values of the linear projection coefficients. (Hint: you will need to compute quantites such as $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

Answer. From the properties of the linear projection, we know $\alpha = \mathbb{E}(Y) - \beta\mathbb{E}(X)$ and $\beta = Cov(X,Y)/Var(X)$.

$$\begin{aligned} \mathbb{E}(X) &= \int_0^1 x f_X(x)\,dx = \int_0^1 x\frac{3}{2}\left(x^2+\frac{1}{3}\right)dx \\ &= \frac{3}{2}\int_0^1 \left(x^3+\frac{1}{3}x\right)dx \\ &= \frac{3}{2}\left[\frac{1}{4}x^4+\frac{1}{6}x^2\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}+\frac{1}{6}\right) \\ &= \frac{3}{2}\left(\frac{5}{12}\right) = \frac{5}{8}. \end{aligned}$$

The expected value of Y is computed in identical fashion, and so $\mathbb{E}(Y) = \frac{5}{8}$.

To compute β, we need to additional moments

$$\begin{aligned}
\mathbb{E}\left(X^2\right) &= \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2 \frac{3}{2}\left(x^2+\frac{1}{3}\right)dx \\
&= \frac{3}{2}\int_0^1 \left(x^4+\frac{1}{3}x^2\right)dx \\
&= \frac{3}{2}\left[\frac{1}{5}x^5+\frac{1}{9}x^3\right]_0^1 = \frac{3}{2}\left(\frac{1}{5}+\frac{1}{9}\right) \\
&= \frac{3}{2}\left(\frac{14}{45}\right) = \frac{7}{15},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(XY) &= \int_0^1\int_0^1 xy f_{X,Y}(x,y)\,dydx = \int_0^1\int_0^1 xy\frac{3}{2}\left(x^2+y^2\right)dydx \\
&= \frac{3}{2}\int_0^1\int_0^1 \left(x^3y+xy^3\right)dydx \\
&= \frac{3}{2}\int_0^1 \left[\frac{1}{2}x^3y^2+\frac{1}{4}xy^4\right]_0^1 dx \\
&= \frac{3}{2}\int_0^1 \left(\frac{1}{2}x^3+\frac{1}{4}x\right)dx \\
&= \frac{3}{2}\left[\frac{1}{8}x^4+\frac{1}{8}x^2\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}.
\end{aligned}$$

We now compute

$$\begin{aligned}
Cov(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) \\
&= \frac{3}{8} - \left(\frac{5}{8}\right)^2 = \frac{24}{64} - \frac{25}{64} = -\frac{1}{64},
\end{aligned}$$

and

$$\begin{aligned}
Var(X) &= \mathbb{E}\left(X^2\right) - (\mathbb{E}X)^2 \\
&= \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{448}{960} - \frac{375}{960} = \frac{73}{960}.
\end{aligned}$$

We compute our final answers for the linear projection coefficients

$$\begin{aligned}
\beta &= \frac{\frac{-1}{64}}{\frac{73}{960}} = -\frac{15}{73}, \\
\alpha &= \frac{5}{8} - \left(-\frac{15}{73}\right)\frac{5}{8} \\
&= \frac{5}{8}\frac{88}{73} = \frac{55}{73}.
\end{aligned}$$

c) Consider the linear projection

$$X_i = \alpha^* + \beta^* Y_i + \varepsilon_i^*.$$

Derive the values of the linear projection coefficients.

Answer. The linear projection coefficients are

$$\alpha^* = \mathbb{E}(X_i) - \beta^* \mathbb{E}(Y_i) \quad \text{and} \quad \beta^* = \frac{Cov(Y_i, X_i)}{Var(Y_i)}.$$

Because $f_X(x)$ and $f_Y(y)$ are identical, $Var(Y) = Var(X)$ and, as already computed, $\mathbb{E}(Y_i) = \mathbb{E}(X_i)$. Hence, $\beta^* = \beta$ and $\alpha^* = \alpha$.

2. Let $\{Y_i\}$ be a sequence of independently distributed Bernoulli random variables.

I) Assume the random variables are identically distributed with $\mathbb{P}(Y_i = 1) \overset{def}{=} p$ and $0 < p < 1$. The Lindberg-Levy CLT states that if the Y_i are independent and identically distributed and $\mathbb{E}(Y_i^2) < \infty$, then

$$\sqrt{n}\left(\overline{Y} - \mu\right) \rightsquigarrow \mathcal{N}\left(0, \sigma^2\right),$$

with $\mu = \mathbb{E}(Y_i)$ and $\sigma^2 = \mathbb{E}(Y_i - \mu)^2$.

a) Determine the values of μ and σ^2 and verify the finite second moment condition.

Answer. Compute the mean and variance

$$\mu = \mathbb{E}(Y_i) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Because $\mathbb{E}\left(Y_i^2\right) = 1$,

$$\sigma^2 = Var\left(Y_i\right) = p - p^2 = p\left(1-p\right).$$

Further, $\mathbb{E}\left(Y_i^2\right) = 1 < \infty$, so the condition obviously holds once the distribution is stated to be Bernoulli.

II) Now assume the random variables are not identically distributed and $\mathbb{P}\left(Y_{ni} = 1\right) = p_n$ and $p_n \in (0,1)$. Assume that $p_n \to p$ as $n \to \infty$. A version of the Lindberg-Feller CLT states that if the Y_{ni} are independent but not necessarily identically distributed with finite means $\mathbb{E}\left(Y_{ni}\right) = \mu_{ni}$ and variances $\mathbb{E}\left(Y_{ni} - \mu_{ni}\right)^2$, and if (condition 1)

$$\sup_{n,i} \mathbb{E}\left|Y_{ni}\right|^{2+\delta} < \infty,$$

and if (condition 2)

$$\overline{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} \sigma_{ni}^2 \to \sigma^2 < \infty,$$

then

$$\sqrt{n}\left(\overline{Y} - \mathbb{E}\left(\overline{Y}\right)\right) \rightsquigarrow \mathcal{N}\left(0, \sigma^2\right).$$

b) Verify conditions 1 and 2.

Answer. For each n, $\mathbb{E}\left|Y_{ni}\right|^{2+\delta} = 1^{2+\delta} p_n < \infty$. Also, for each n, $\overline{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} p_n\left(1-p_n\right)$, so $p_n \to p$ implies $\overline{\sigma}_n^2 \to p\left(1-p\right)$ by continuity.

III) Now assume that the random variables are not identically distributed and $\mathbb{P}\left(Y_{ni} = 1\right) = p_{ni}$ and $0 < c \leq p_{ni} \leq d < 1$. A version of the Lindberg-Feller CLT states that if the Y_{ni} are independent but not necessarily identically distributed with finite means $\mathbb{E}\left(Y_{ni}\right) = \mu_{ni}$ and variances $\mathbb{E}\left(Y_{ni} - \mu_{ni}\right)^2$, and if (condition 1)

$$\sup_{n,i} \mathbb{E}\left|Y_{ni}\right|^{2+\delta} < \infty,$$

and if (condition 2)

$$\liminf_{n\to\infty} \overline{\sigma}_n^2 > 0,$$

then

$$\frac{\sqrt{n}\left(\overline{Y} - \mathbb{E}\left(\overline{Y}\right)\right)}{\overline{\sigma}_n} \rightsquigarrow \mathcal{N}\left(0, 1\right).$$

c) Verify conditions 1 and 2.

Answer. For each n and each i, $\mathbb{E}\left|Y_{ni}\right|^{2+\delta} = 1^{2+\delta}p_{ni} < \infty$. For each n, $\overline{\sigma}_n^2 > \min\left\{c\left(1-c\right), d\left(1-d\right)\right\}$, so $\liminf_{n\to\infty}\overline{\sigma}_n^2 = \min\left\{c\left(1-c\right), d\left(1-d\right)\right\} > 0$.