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# Section 5

**Problem 1.** Consider the standard growth model with an infinitely-lived, representative household whose preferences can be summarized by a utility function $u(c_t)$, where $c_t$ is consumption (you may assume that $u$ is increasing and concave). The household maximizes the present discounted value of lifetime utility, where the future is discounted at some factor $\beta \in (0,1)$. Households inelastically supply 1 unit of labor to earn income $w_t$ and can save $(k_{t+1})$, commanding some interest rate $r_{t+1}$.

Firms hire labor and capital services from households, and production takes place according to a CRS production function $f(k_t)$ that satisfies the usual properties. Assume that there is full depreciation ($\delta = 1$) and that initial capital, $k_0$, is given. That is, the household's budget constraint is

$$c_t + k_{t+1} = w_t + r_t k_t.$$

**(a)** Write down the problem's faced by households and firms *sequentially*.

Households:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{s.t.} \qquad \begin{aligned} & c_t + k_{t+1} = w_t + r_t k_t \ \forall t \\ & c_t,\ k_t \geq 0 \ \forall t \\ & k_0 \text{ given} \end{aligned}$$

Firms:

$$\max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \Big[ f(k_t) - r_t k_t - w_t \Big] \qquad \text{s.t.} \qquad k_t \geq 0 \ \forall t$$

**(b)** Define a *sequential markets equilibrium* in this environment.

A SME consists of prices $\{r_t, w_t\}_{t=0}^{\infty}$, allocations for households $\{c_t, k_t^d\}_{t=0}^{\infty}$, and allocations for firms $\{k_t^s\}_{t=0}^{\infty}$ such that the following holds.

1. Given prices, the allocation of the representative firm solves the firm's problem.
2. Given prices, the allocation of the representative household solves the household's problem.

3. Markets clear:

$$f(k_t) = c_t + k_{t+1} \qquad \text{(goods market)}$$
$$k_t^s = k_t^s. \qquad \text{(capital market)}$$

**(c)** Write down the problem of the representative household *recursively.*

$$V(k, K) = \max_{c,k'} \left\{ u(c) + \beta V(k', K') \right\} \qquad \text{s.t.} \qquad c + k' = w + rk$$
$$c,\ k \geq 0$$
$$K' = \phi(K)$$

[10] **(d)** Define a *recursive competitive equilibrium* in this environment.

A RCE is a set of functions that describe

Quantities: $k' = g(k, K)$, $K' = G(K)$
Lifetime utility: $V(k, K)$
Prices: $r(K) = f'(K)$, $w(K) = f(K) - Kf'(K)$

such that the following hold.

1. prices are complete and given.
2. $V(k, K)$ and $g(k, K)$ solve the household's problem.
3. There is consistency: $G(K) = g(K, K)$.

**Problem 2.** *Regular and Overtime Hours of Work.* Consider an economy that is comprised of a continuum of infinitely-lived individuals each of whom work one of two types of work shifts or not at all. The shifts correspond to working only a straight time shift, $(h_1)$, or straight time plus overtime $(h_1 + h_2)$. Let $n_{1t}$ be the fraction of individuals working only straight time and $n_{2t}$ be the fraction of individuals working straight time plus overtime. Each individual has preferences given by

$$E \sum_{t=0}^{\infty} [\log c_t + A \log(1 - h_t)],$$

where $h_t \in \{0, h_1, h_1 + h_2\}$.

Output can be used for consumption, $c$, or investment, $i$, and is produced using a Cobb-Douglas technology given by:

$$y_t = e^{z_t} K_t^{\theta} (H_{1t}^{1-\theta} + H_{2t}^{1-\theta}),$$

where $K_t$ is the capital stock, $H_{1t}$ is total straight time hours worked and $H_{2t}$ is total overtime hours worked. Investment in period $t$ becomes productive in period $t+1$ and the stock of capital depreciates at rate $\delta$. The technology shock evolves according to a first order autoregressive process:

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \text{where } \epsilon \sim N(0, \sigma^2).$$

Hint: This is NOT a heterogeneous agent question.

**(a)** Write the social planner's problem as a dynamic program where the planner gives equal weight to the utility of all individuals.

The state variables in the planners problem are $z$ and $k$. The choice variables are $c$, $x$, $n_1$ and $n_2$. We can write the planners problem as follows,

$$V(z,k) = \max_{c,x,n_1,n_2} \left\{ \log(c) + n_1 A \log(1-h_1) + n_2 A \log(1-h_1-h_2) + \beta E[V(z',k')|z] \right\}$$

subject to

$$\begin{aligned} c + x &= e^z k^\theta (H_1^{1-\theta} + H_2^{1-\theta}) \\ H_1 &= (n_1+n_2)h_1 \\ H_2 &= n_2 h_2 \\ k' &= (1-\delta)k + x \\ z' &= \rho z + \varepsilon \\ c, x &\geq 0 \\ n_1 + n_2 &\in [0,1]. \end{aligned}$$

**(b)** Define a *recursive competitive equilibrium* for this economy. Be specific about the market structure assumed.

A recursive competitive equilibrium in this economy is a value function $V(z,k,K)$, individual decision rules $g_c(z,k,K)$, $g_x(z,k,K)$, $g_{n_1}(z,k,K)$, $g_{n_2}(z,k,K)$, aggregate decision rules $G_C(z,K)$,$G_X(z,K)$,$G_{N_1}(z,K)$, $G_{N_2}(z,K)$, pricing functions $r(z,K)$, $w_1(z,K)$, $w_2(z,K)$, and an aggregate law of motion $K'(z,K)$ such that the following conditions are satisfied,

1. (Household's Problem) Taking prices as given, the value function and policy functions solve,

$$\begin{aligned} V(z,k,K) = \max_{c,x,n_1,n_2} \{&\log(c) + n_1 A \log(1-h_1) + n_2 A \log(1-h_1-h_2) \\ &+ \beta E[V(z',k',K')|z]\} \end{aligned}$$

subject to

$$
\begin{aligned}
c + x &= r(z,K)k + (n_1+n_2)w_1(z,K)h_1 + n_2 w_2(z,K)h_2 \\
C &= G_C(z,K) \\
X &= G_X(z,K) \\
N_1 &= G_{N_1}(z,K) \\
N_2 &= G_{N_2}(z,K) \\
k' &= (1-\delta)k + x \\
K' &= (1-\delta)K + X \\
c, x &\geq 0 \\
n_1 + n_2 &\in [0,1].
\end{aligned}
$$

2. (Firm's Problem)

$$
\begin{aligned}
r(z,K) &= \theta e^z K^{\theta-1}(H_1^{1-\theta} + H_2^{1-\theta}) \\
w_1(z,K) &= (1-\theta)e^z K^{\theta} H_1^{-\theta} \\
w_2(z,K) &= (1-\theta)e^z K^{\theta} H_2^{-\theta}.
\end{aligned}
$$

3. (Consistency) $g_c(z,K,K) = G_C(z,K), g_x(z,K,K) = G_X(z,K), g_{n_1}(z,K,K) = G_{N_1}(z,K), g_{n_2}(z,K,K) = G_{N_2}(z,K)$.

4. (Market Clearing) Labor Market Clears, $H_1 = (N_1+N_2)h_1$ and $H_2 = N_2 h_2$. Goods market clears through budget constraint and aggregate consistency conditions. The condition would be $C + X = Y$.